Section I
10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

NESA Number: | | | | | | | | |

Use the multiple-choice answer sheet (*provided on the last page of the booklet*) for Questions 1–10.

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1. What is the scalar product of the two vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{v} = 4\underset{\sim}{i} - 6\underset{\sim}{j} - 3\underset{\sim}{k}$?

A. 25

B. -11

C. 5

D. 23

2. It is given that a, b, c and d are consecutive integers.

Which of the following statements may be false?

A. $abcd$ is divisible by 8

B. $abcd$ is divisible by 3

C. $a + b + c + d$ is divisible by 4

D. $a + b + c + d$ is divisible by 2

3. What is the contrapositive of the following statement?

If you're sad and you know it, then you will stomp your feet.

A. If you don't stomp your feet, then you're sad and you know it.

B. If you stomp your feet, then you're either not sad or you don't know it.

C. If you don't stomp your feet, then you're not sad and you don't know it.

D. If you don't stomp your feet, then you're either not sad or you don't know it.

4. Which of the following is an expression for $\int \frac{x}{\sqrt[3]{x^2+1}}\,dx$?

A. $\frac{3}{2}\sqrt[3]{(x^2+1)^2}+C$

B. $\frac{3}{4}\sqrt[3]{(x^2+1)^2}+C$

C. $\frac{1}{4}\sqrt[3]{(x^2+1)^2}+C$

D. $\frac{2}{3}\sqrt[3]{(x^2+1)^2}+C$

5. $(*)$ A whole number n is prime if it is 1 less or 5 less than a multiple of 6. How many counterexamples to $(*)$ are there in the range $0 < n < 50$?

A. 2

B. 3

C. 4

D. 5

6. If $z = (1+ia)^2$ where a is real and positive, what is the exact value of a if $\arg(z) = \frac{\pi}{3}$?

A. $\frac{\pi}{6}$

B. $\frac{1}{\sqrt{3}}$

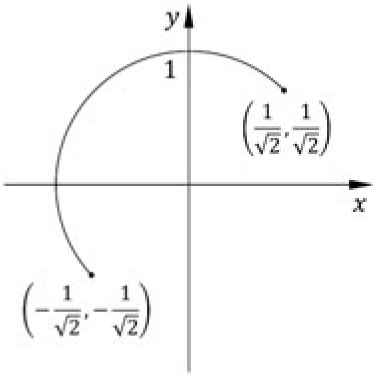
C. $\sqrt{3}$

D. $\frac{1}{3}$

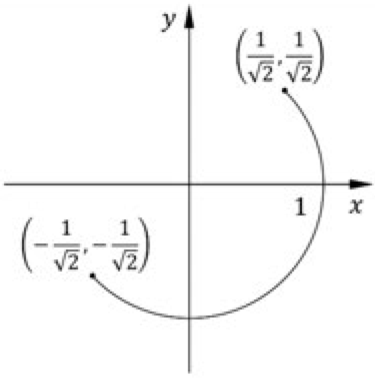
7. Which diagram best shows the curve described by the position vector

$$\underset{\sim}{r}(t) = \sin(t)\,\underset{\sim}{i} - \cos(t)\,\underset{\sim}{j} \ \text{ for } \frac{\pi}{4} \le t \le \frac{5\pi}{4}\ ?$$

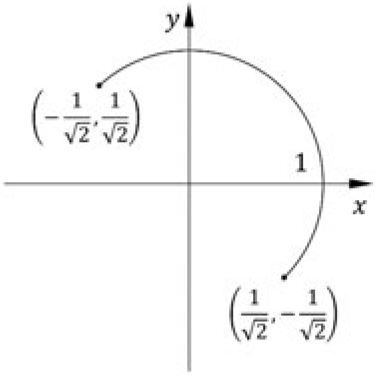
A.



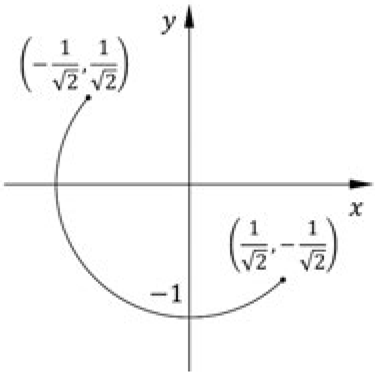
B.



C.



D.



8. On an Argand diagram, the points $z, -\bar{z}, z^{-1}$ and $-\overline{(z^{-1})}$, where $|z| \neq 1$, form the vertices of a shape.

Which of the following is the shape?

A. Square

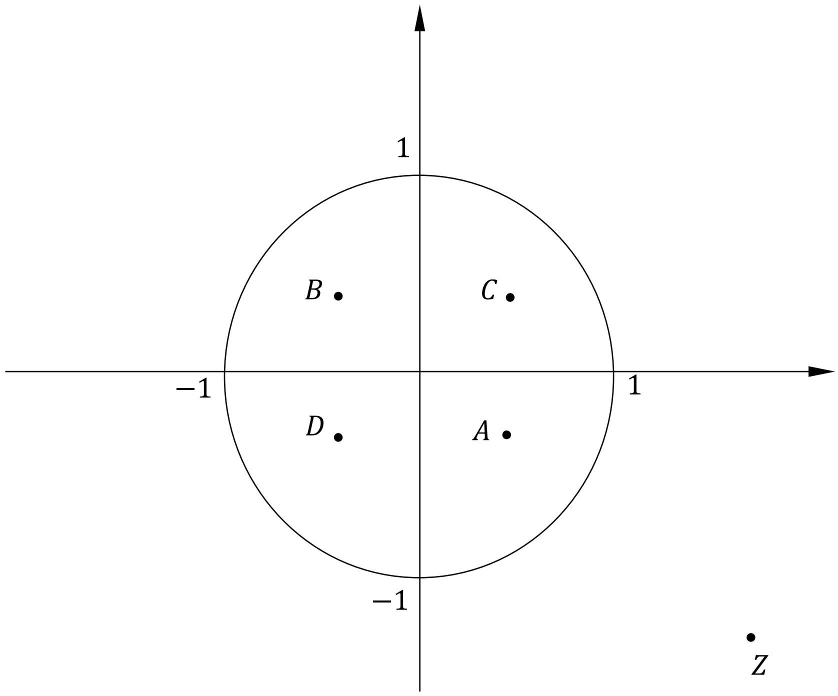
B. Rectangle

C. Rhombus

D. Trapezium

9. The diagram shows the complex number z in the fourth quadrant of the complex plane. The modulus of Z is 2.

Which of the points marked $A,\ B,\ C$ or D best shows the position of $-\dfrac{1}{iZ}$?



A. Point A

B. Point B

C. Point C

D. Point D

10. If $\omega = \cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}$, which quartic polynomial has $\omega, \omega^3, \omega^7$ and ω^9 as its zeros?

A. $z^4 + z^3 + z^2 + z + 1$

B. $z^4 - z^3 + z^2 - z + 1$

C. $z^4 - z^3 - z^2 + z + 1$

D. $z^4 + z^3 - z^2 - z + 1$

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use a new writing booklet.

(a) The complex numbers $z = 9e^{\frac{\pi}{3}i}$ and $w = 3e^{\frac{\pi}{6}i}$ are given.

(i) Find the value of $\frac{z}{w}$, giving the answer in the form $re^{i\theta}$. **1**

(ii) Hence, or otherwise, find the value of w^2. **1**

(b) It is given that the point R is $(2,1,-1)$, $\overrightarrow{RS} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$ and $\overrightarrow{RT} = 3\overrightarrow{RS}$. **2**

Find the coordinates of T.

(c) Find

(i) $\displaystyle\int \sin^3 x \, dx$ **2**

(ii) $\displaystyle\int \frac{x^2}{x^6+6} \, dx$ **2**

(d) $2-3i$ is one root of the equation $z^3 + mz + 52 = 0$, where m is real.

(i) Find the other roots. **2**

(ii) Determine the value of m. **2**

(e) (i) Find the square roots of $-3-4i$. **2**

(ii) Hence or otherwise, solve the equation $z^2 - 3z + (3+i) = 0$. **2**

Question 12 (14 marks) Use a separate writing booklet.

(a) (i) Express $\dfrac{3x^2-3x+5}{x\left(x^2+5\right)}$ as a sum of partial fractions over $\mathbb{R}$. **3**

(ii) Hence find $\displaystyle\int \frac{3x^2-3x+5}{x\left(x^2+5\right)}\,dx$. **2**

(b) For all non-negative numbers, x and y, $\dfrac{x+y}{2} \geq \sqrt{xy}$. (Do NOT prove this.) **2**

A rectangle has dimensions a and b.

Given that the rectangle has perimeter P, and area A, prove that $P^2 \geq 16A$.

(c) (i) Write the complex number $w=\dfrac{8-2i}{5+3i}$ in the form $x+iy$ **1**

(ii) Find the argument of w. **1**

(iii) Hence or otherwise, find the possible values of the positive integer n for which w^n is purely real. **2**

(d) On an Argand diagram, sketch the region satisfied by both **3**

$$|z+1| \leq |z-i| \text{ and } \operatorname{Im}(z) < 2.$$

Question 13 (14 marks) Use a separate writing booklet.

(a) A triangle has side lengths $x+3$, $3x+6$ and $5x+2$, where $x \in \mathbb{R}$. **2**

Prove that $\frac{1}{3} < x < 7$.

(b) Suppose $p \in \mathbb{R}$ satisfies $7^p = 2$. Prove that p is irrational. **2**

(c) Let a_n be the sequence defined recursively by $a_0 = 0$ and $a_n = a_{n-1} + 3n^2$ for all integers $n \geq 1$. **3**

Use mathematical induction to prove that for all integers $n \geq 0$,

$$a_n = \frac{n(n+1)(2n+1)}{2}$$

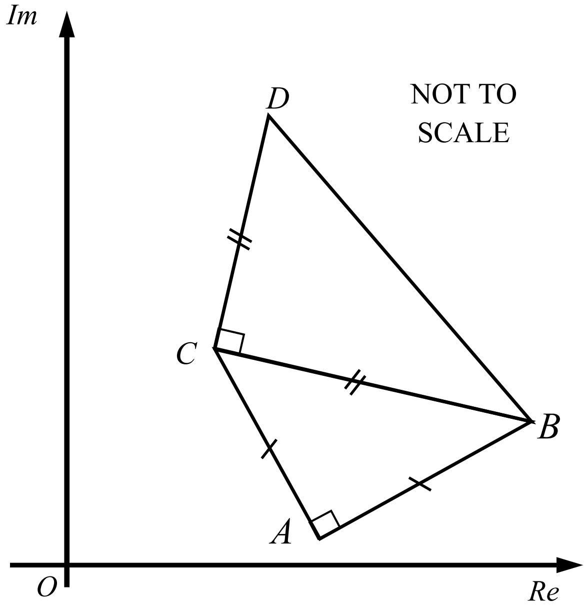
(d) A quadrilateral is formed in three-dimensional space. **4**

It's vertices are $O(0,0,0)$, $A(2,5,-6)$, $B(3,-3,-4)$ and $C(2,-16,4)$, labelled in the clockwise direction from point O.

Find the size of $\angle ABC$, giving your answer to the nearest degree.

Question 13 continues on page 8

(e)



In the diagram the points A, B, C and D represent the complex numbers z_1, z_2, z_3 and z_4, respectively. Both ΔABC and ΔBCD are right angled isosceles triangles as shown

(i) Show that the complex number z_3 can be written as **1**

$$z_3 = (1-i)z_1 + iz_2$$

(ii) Hence, express the complex number z_4 in terms of z_1 and z_2, giving your answer in simplest form. **2**

Question 14. (14 marks) Use a separate writing booklet.

(a) By considering the series $\sum_{k=1}^{n} k$ and the AM-GM Inequality $\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$, **3**

prove that $\left(\frac{n+1}{2}\right)^n \geq n!$ for integers $n \geq 1$.

(b) Use integration by parts to evaluate $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}}\, dx$. **3**

(c) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate **3**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2}{\sin x + 1}\, dx$$

(d) Given $|z - 2 - 2i| = 1$.

(i) On an argand diagram, sketch the graph of the set of points represented by z. **1**

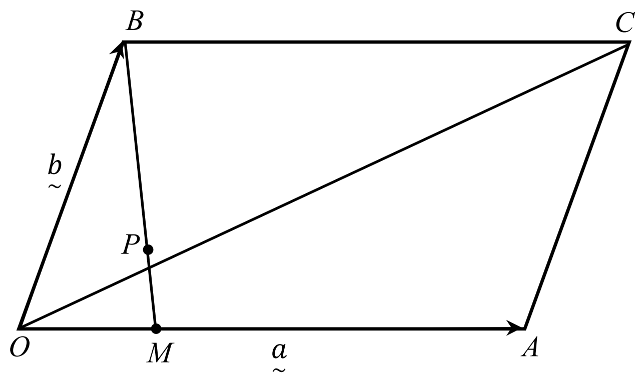
(ii) Find the maximum value of Arg z. **2**

(iii) Find the maximum value of $|z|$. **2**

Question 15. (16 marks) Use a separate writing booklet.

(a) $OACB$ is a parallelogram with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. M is a point on OA such that $\left|\overrightarrow{OM}\right| = \frac{1}{5}\left|\overrightarrow{OA}\right|$. P is a point on MB such that $\left|\overrightarrow{MP}\right| = \frac{1}{6}\left|\overrightarrow{MB}\right|$, as shown in the diagram. **3**

Show that P lies on OC.



(b) (i) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **1**

(ii) Show that: **2**

$$\int_0^1 \frac{dx}{x+\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}d\theta$$

(iii) Hence, determine the value of: **2**

$$I = \int_0^1 \frac{dx}{x+\sqrt{1-x^2}}$$

Question 15 continues on page 11

(c) The line $\underset{\sim}{r}_1$ has equation:

$$\underset{\sim}{r}_1 = \underset{\sim}{i} + 2\underset{\sim}{k} + \lambda\left(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}\right) \quad \text{where } \lambda \in \mathbb{R}.$$

The line $\underset{\sim}{r}_2$ has equation:

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-1}{3}$$

(i) Show that $\underset{\sim}{r}_2 = -\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k} + \mu\left(\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$ where $\mu \in \mathbb{R}$. **1**

(ii) Show that lines $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ do not intersect. **3**

The point A lies on $\underset{\sim}{r}_1$ with parameter $\lambda = p$, and the point B lies on $\underset{\sim}{r}_2$ with parameter $\mu = q$.

(iii) Write $\overrightarrow{AB}$ as a column vector. **1**

(iv) Calculate the value of $\left|\overrightarrow{AB}\right|$ when $\overrightarrow{AB}$ is perpendicular to both $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$. **3**

Question 16. (16 marks) Use a separate writing booklet.

(a) Consider the proposition:

If the remainder is 2 or 3 when an integer n *is divided by 4, then* $n \neq k^2$*, where* $k \in \mathbb{Z}$.

(i) State the contrapositive to the proposition. **1**

(ii) Hence, prove the proposition by proving the contrapositive. **3**

(b) Let

$$z_n = \frac{1}{(1+i)^0} + \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^n}$$

(i) Express $\dfrac{1}{1+i}$ in the form $x + iy$. **1**

(ii) Prove that $z_n = 1 - i + \dfrac{\sin\frac{\pi n}{4} + i\cos\frac{\pi n}{4}}{2^{\frac{n}{2}}}$. **4**

(c) Let $I_n = \int_0^a x^n \sqrt{a^2 - x^2}\, dx$, $a \in \mathbb{R}^+$ and $n = 0, 1, \ldots$

(i) Prove that $I_n = a^2 \dfrac{n-1}{n+2} I_{n-2}$ for $n = 2, 3, \ldots$ **3**

(ii) Prove that $I_{2n} = \pi \left(\dfrac{a}{2}\right)^{2n+2} \dfrac{(2n)!}{n!(n+1)!}$ **4**

End of examination.

Fort Street High School

NESA Number:

Name:

Teacher:	Ms Hussein	Ms Kaur	Mr Moon

2023

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

SOLUTIONS

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Approved calculators may be used
- A reference sheet is provided
- Marks may be deducted for careless or badly arranged work.
- In Questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks : 100

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks

- Allow about 2 hours and 45 minutes for this section
- Write your student number on each answer booklet.
- Attempt Questions 11 – 16

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	2 + 4 =	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ● C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A ● B ● correct C ○ D ○

1.	A ○	B ○	C ●	D ○
2.	A ○	B ○	C ●	D ○
3.	A ○	B ○	C ○	D ●
4.	A ○	B ●	C ○	D ○
5.	A ○	B ○	C ●	D ○
6.	A ○	B ●	C ○	D ○
7.	A ○	B ○	C ●	D ○
8.	A ○	B ○	C ○	D ●
9.	A ○	B ●	C ○	D ○
10.	A ○	B ●	C ○	D ○

Section I

NESA Number: | | | | | | | | |

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet (*provided on the last page of the booklet*) for Questions 1–10.

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1. What is the scalar product of the two vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{v} = 4\underset{\sim}{i} - 6\underset{\sim}{j} - 3\underset{\sim}{k}$?

A. 25

B. 11

C. 5

D. 23

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -6 \\ -3 \end{bmatrix} = 8+6-9$$
$$= 5$$

2. It is given that a, b, c and d are consecutive integers.

Which of the following statements may be false?

A. $abcd$ is divisible by 8

B. $abcd$ is divisible by 3

C. $a + b + c + d$ is divisible by 4

D. $a + b + c + d$ is divisible by 2

Take 1, 2, 3, 4
$1+2+3+4 = 10$ (÷ by 2)
$1 \times 2 \times 3 \times 4 = 24$ (÷ by 8 and by 3)
The only statement that doesn't work is C.

3. What is the contrapositive of the following statement?

If you're sad and you know it, then you will stomp your feet.

A. If you don't stomp your feet, then you're sad and you know it.

B. If you stomp your feet, then you're either not sad or you don't know it.

C. If you don't stomp your feet, then you're not sad and you don't know it.

D. If you don't stomp your feet, then you're either not sad or you don't know it.

4. Which of the following is an expression for $\int \frac{x}{\sqrt[3]{x^2+1}}\,dx$?

A. $\frac{3}{2}\sqrt[3]{(x^2+1)^2}+C$

B. $\frac{3}{4}\sqrt[3]{(x^2+1)^2}+C$

C. $\frac{1}{4}\sqrt[3]{(x^2+1)^2}+C$

D. $\frac{2}{3}\sqrt[3]{(x^2+1)^2}+C$

5. $(*)$ A whole number n is prime if it is 1 less or 5 less than a multiple of 6. How many counterexamples to $(*)$ are there in the range $0 < n < 50$?

A. 2

B. 3

C. 4

D. 5

Multiples of 6	**1 less**	**5 less**	**Prime**
6	**5**	**1**	**N**
12	**11**	**7**	**Y**
18	**17**	**13**	**Y**
24	**23**	**19**	**Y**
30	**29**	**35**	**N**
36	**35**	**31**	**N**
42	**41**	**37**	**Y**
48	**47**	**43**	**Y**
54	**Not < 50**	**49**	**N**

6. If $z = (1 + ia)^2$ where a is real and positive, what is the exact value of a if $\arg(z) = \dfrac{\pi}{3}$?

A. $\dfrac{\pi}{6}$

B. $\dfrac{1}{\sqrt{3}}$

C. $\sqrt{3}$

D. $\dfrac{1}{3}$

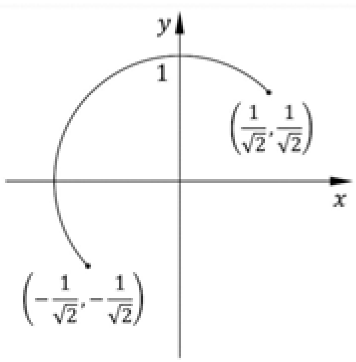
$\arg(z) = \arg[(1+ia)^2]$
$= 2\arg(1+ia)$
$\arg(1+ia) = \frac{\pi}{6}$
$\tan^{-1}\left(\frac{a}{1}\right) = \frac{\pi}{6}$
$\therefore a = \frac{1}{\sqrt{3}}$

7. Which diagram best shows the curve described by the position vector

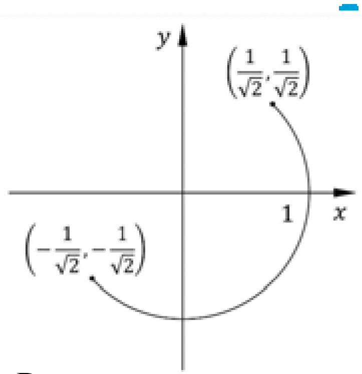
$$\underset{\sim}{r}(t) = \sin(t)\underset{\sim}{i} - \cos(t)\underset{\sim}{j} \text{ for } \frac{\pi}{4} \le t \le \frac{5\pi}{4}?$$

$\underset{\sim}{r}\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4}\underset{\sim}{i} - \cos\frac{\pi}{4}\underset{\sim}{j}$
$= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
$\underset{\sim}{r}\left(\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$\underset{\sim}{r}\left(\frac{\pi}{2}\right) = (1, 0)$

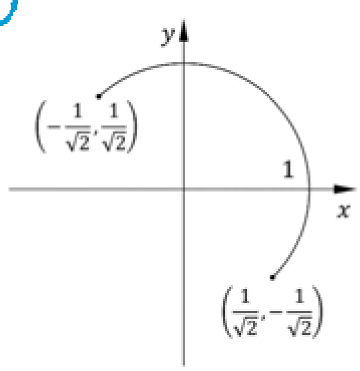
A.



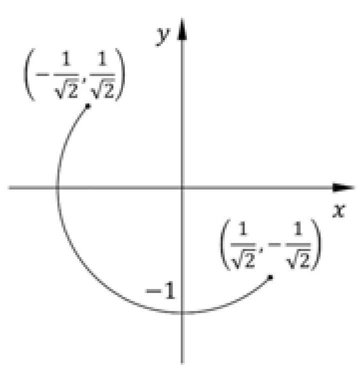
B.



C.



D.



8. On an Argand diagram, the points $z, -\underline{z}, z^{-1}$ and $-\overline{(z^{-1})}$, where $|z| \neq 1$, form the vertices of a shape.

Consider $z = x + iy$

Which of the following is the shape?

$-\bar{z} = -(x - iy)$
$= -x + iy \Rightarrow$ a reflection in y axis

A. Square

B. Rectangle

C. Rhombus

D. Trapezium

$z^{-1} = \frac{1}{x+iy}$

$= \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

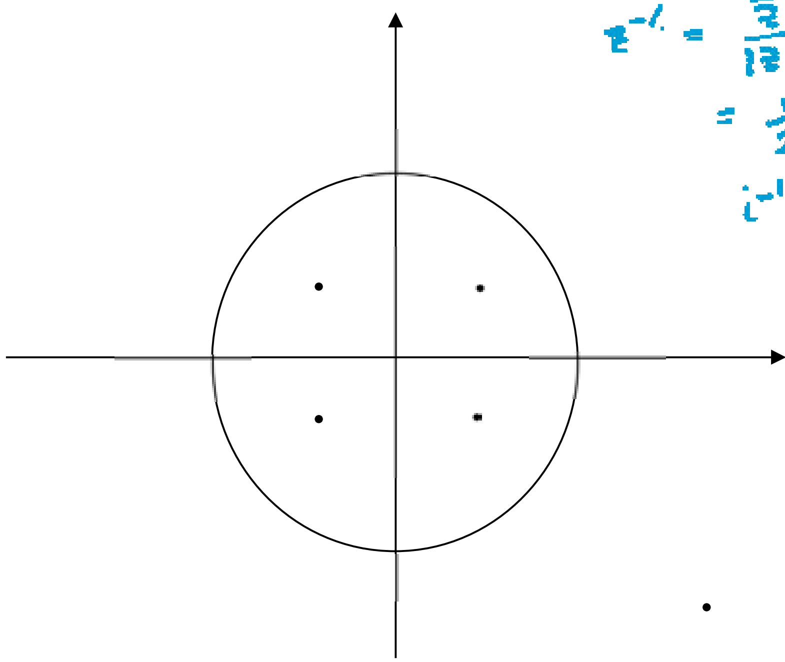
$= \frac{x-iy}{x^2+y^2} \Rightarrow$ reflection in x divided by $|z|^2$

$-\overline{(z^{-1})} = \frac{-x-iy}{x^2+y^2}$

which is a rotation of $180°$ about the origin divided by $|z|^2$

9. The diagram shows the complex number z in the fourth quadrant of the complex plane. The modulus of Z is 2.

Which of the points marked A, B, C or D best shows the position of $-\dfrac{1}{iZ}$?



$z^{-1} = \dfrac{\bar{z}}{|z|}$

$= \frac{1}{2}\bar{z}$ (takes it to C)

i^{-1} = clockwise rotation of $\frac{\pi}{2}$ (takes it to A)

$-i^{-1}$ = anticlockwise rotation of π

A. Point A

B. Point B

C. Point C

D. Point D

10. If $\omega = \cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}$, which quartic polynomial has $\omega, \omega^3, \omega^7$ and ω^9 as its zeros?

A. $z^4 + z^3 + z^2 + z + 1$

B. $z^4 - z^3 + z^2 - z + 1$

C. $z^4 - z^3 - z^2 + z + 1$

D. $z^4 + z^3 - z^2 - z + 1$

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use a new writing booklet.

(a)	The complex numbers $z = 9e^{\frac{\pi}{3}i}$ and $w = 3e^{\frac{\pi}{6}i}$ are given.		
	(i)	Find the value of $\frac{z}{w}$, giving the answer in the form $re^{i\theta}$.	**1**

$$\frac{z}{w} = \frac{9e^{\pi/3 i}}{3e^{\pi/6 i}}$$

$$= 3e^{\pi/6 i}$$

$$= w$$

Marker's comments:
Generally answered well

	(ii)	Hence, or otherwise, find the value of w^2.	**1**

$$\frac{z}{w} = w$$

$$z = w^2$$

$$\therefore w^2 = 9e^{\pi/3 i}$$

Marker's comments:
Generally answered well

(b)	It is given that the point R is $(2,1,-1)$, $\overrightarrow{RS}=\begin{pmatrix}-4\\-1\\2\end{pmatrix}$ and $\overrightarrow{RT}=3\overrightarrow{RS}$. Find the coordinates of T.	**2**

$\overrightarrow{RT} = 3\begin{bmatrix}-4\\-1\\2\end{bmatrix} = \begin{bmatrix}-12\\-3\\6\end{bmatrix}$

$\overrightarrow{RT} = \overrightarrow{OT} - \overrightarrow{OR}$

$\overrightarrow{OT} = \overrightarrow{RT} + \overrightarrow{OR}$

$= \begin{bmatrix}-12\\-3\\6\end{bmatrix} + \begin{bmatrix}2\\1\\-1\end{bmatrix}$

$= \begin{bmatrix}-10\\-2\\5\end{bmatrix}$ $\therefore$ Coordinates of T are $(-10,-2,5)$

Marker's comments:
Generally answered well but students were making careless errors.

(c)	Find		
	(i)	$\int \sin^3 x\,dx$	**2**

$\int \sin^3 x\,dx = \int \sin x \sin^2 x\,dx$

$= \int \sin x(1-\cos^2 x)\,dx$

$= \int \sin x - \cos^2 x \frac{d(-\cos x)}{dx}\,dx$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

Marker's comments:
Generally answered well but students were making careless errors.

Alternative let $u = \cos x$

$\int \sin x\,(1-\cos^2 x)dx = \int u^2 - 1\,du$

$= \frac{u^3}{3} - u + C$

$= \frac{\cos^3 x}{3} - \cos x + C$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$-du = \sin x\,dx$

	(ii)	$\int \frac{x^2}{x^6+6}dx$	**2**

$\frac{1}{3}\int \frac{du}{u^2+6}$

let $u = x^3$

$\frac{du}{dx} = 3x^2$

$\frac{du}{3} = x^2 dx$

$= \frac{1}{3} \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{6}}\right) + C$

$= \frac{\sqrt{6}}{18} \tan^{-1}\left(\frac{x}{\sqrt{6}}\right) + C$

Marker's comments:
Generally answered well but some students had trouble recognising that the integral was arctan or they used the wrong substitution.

(d)	$2-3i$ is one root of the equation $z^3 + mz + 52 = 0$, where m is real.		
	(i)	Find the other roots.	**2**

Since coefficients are real then one of the roots is the conjugate of $2+3i$ ie, $2-3i$

Let 3rd root be α

Sum of roots: $2+3i+2-3i+\alpha = 0$

$4+\alpha = 0$ so $\alpha = -4$

$\therefore$ The other 2 roots are $2+3i$ and -4.

Marker's comments:
Generally answered well but a few students didn't realise that the conjugate was also a root

	(ii)	Determine the value of m.	**2**

Sum of pairs of roots

$\alpha\beta + \alpha\gamma + \beta\gamma = m$

$-4(2+3i) - 4(2-3i) + (2+3i)(2-3i) = m$

$-8-12i-8+12i+13 = m$

$\therefore m = -3$

Marker's comments:
Generally answered well but students were making careless errors.

(e)	(i)	Find the square roots of $-3-4i$.	2

let $z = x+iy$

$z^2 = -3-4i$

$(x+iy)^2 = -3-4i$

$x^2 + 2xyi - y^2 = -3-4i$

$x^2 - y^2 = -3$ —— (1)

$x^2 + y^2 = \sqrt{(-3)^2 + (4)^2}$

$x^2 + y^2 = 5$ —— (2)

$2x^2 = 2$ $\quad \therefore x = \pm 1$

$y = \mp 2$

$\therefore$ Square roots are $z = 1-2i$ and $z = -1+2i$

Marker's comments:
Generally answered well but methods used to answer the question were not always the most efficient.

	(ii)	Hence or otherwise, solve the equation $z^2 - 3z + (3+i) = 0$.	2

$$z = \frac{3 \pm \sqrt{9-4(3+i)}}{2}$$

$$= \frac{3 \pm \sqrt{-3-4i}}{2}$$

from part (i)

$$z = \frac{3+1-2i}{2} = 2-i \qquad z = \frac{3-1+2i}{2} = 1+i$$

Marker's comments:
Generally answered well.

Question 12 (14 marks) Use a separate writing booklet.

(a) (i) Express $\dfrac{3x^2-3x+5}{x\left(x^2+5\right)}$ as a sum of partial fractions over $\mathbb{R}$. **3**

a) i) $\dfrac{3x^2-3x+5}{x(x^2+5)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+5}$ ✓

$3x^2-3x+5 = A(x^2+5) + x(Bx+C)$

let $x=0 \Rightarrow 5 = 5A \quad \therefore A=1$

let $x=1 \Rightarrow 5 = 6 + B + C$

$B+C=-1$ — (1)

let $x=-1 \Rightarrow 11 = 6 + B - C$

$B-C=5$ — (2)

$2B=4 \quad \therefore \underline{B=2}$

$2-C=5 \quad \therefore \underline{C=-3}$

$\therefore A=1 \quad B=2 \quad C=-3$ ✓

Progress ✓

Marker's comments:
Generally answered well although students made careless errors in their solutions.

(ii) Hence find $\displaystyle\int \frac{3x^2-3x+5}{x\left(x^2+5\right)}\,dx$. **2**

ii) $\displaystyle\int \frac{3x^2-3x+5}{x(x^2+5)}\,dx = \int \frac{1}{x} + \frac{2x-3}{x^2+5}\,dx$

$= \displaystyle\int \frac{1}{x} + \frac{2x}{x^2+5} - \frac{3}{x^2+5}\,dx$ ✓

$= \ln|x| + \ln|x^2+5| - \frac{3}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$

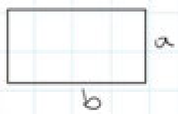
$= \ln|x(x^2+5)| - \frac{3}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$ ✓

Marker's comments:
Generally answered well.

(b) For all non-negative numbers, x and y, $\frac{x+y}{2} \geq \sqrt{xy}$. (Do NOT prove this.) **2**

A rectangle has dimensions a and b.

Given that the rectangle has perimeter P, and area A, prove that $P^2 \geq 16A$.



$2a + 2b = P$ — (1)

$A = ab$ — (2)

From (1) $a + b = \frac{P}{2}$

But $\frac{a+b}{2} \geqslant \sqrt{ab}$ given

So, $a + b \geqslant 2\sqrt{ab}$

{using (1)} $\frac{P}{2} \geqslant 2\sqrt{A}$

$\frac{P^2}{4} \geqslant 4A$

$\therefore P^2 \geqslant 16A$ as req'd

Marker's comments:
Most students were able to arrive at the proof, however, the setting out and structure of the logic was an issue.

(c) (i) Write the complex number $w = \frac{8-2i}{5+3i}$ in the form $x + iy$ **1**

i) $w = \frac{8-2i}{5+3i} \times \frac{5-3i}{5-3i}$

$= \frac{40 - 34i - 6}{25 + 9}$

$= \frac{34 - 34i}{34}$

$\therefore w = 1 - i$

Marker's comments:
Generally answered well

(ii) Find the argument of w. **1**

ii) $\arg(w) = \tan^{-1}(-1)$

$= -\frac{\pi}{4}$

Marker's comments:
Generally answered well

(iii) Hence or otherwise, find the possible values of the positive integer n for which w^n is purely real. **2**

iii) $w^n = (\sqrt{2}\,\text{cis}(-\frac{\pi}{4}))^n$

By DMT: $w^n = (\sqrt{2})^n\,\text{cis}(-\frac{n\pi}{4})$

If purely real, then

$\text{cis}(-\frac{n\pi}{4}) = \text{cis}(k\pi)$, where $k \in \mathbb{Z}$

Hence,

$-\frac{n\pi}{4} = k\pi$

$\therefore \underline{n = -4k}$ since n is positive then $k<0$

Must include $k<0$ or equivalent

Marker's comments:
Many students forgot to qualify the sign of k to allow for positive values only of n.

(d) On an Argand diagram, sketch the region satisfied by both **3**

$$|z+1| \le |z-i| \text{ and } \operatorname{Im}(z) < 2.$$

let $z = x+iy$

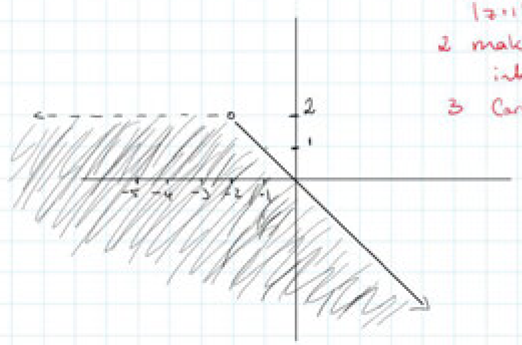
$|x+1+iy| \le |x+iy-i|$

$(x+1)^2+y^2 \le x^2+(y-1)^2$

$x^2+2x+1+y^2 \le x^2+y^2-2y+1$

$\therefore y \le -x$

1 correctly sketches 1 of $|z+1| \le |z-i|$ or $\operatorname{Im}(z) < 2$
2 makes progress sketching intersection
3 Correct solution



Marker's comments:
Students were not:

- including the open circle at the intersection point of the two graphs
- did not include a dashed line beyond the applicable region, however, marks were not deducted for this
- using the correct Cartesian equation for $|z+1| \le |z-i|$

Question 13 (14 marks) Use a separate writing booklet.

(a) A triangle has side lengths $x+3$, $3x+6$ and $5x+2$, where $x \in \mathbb{R}$. **2**

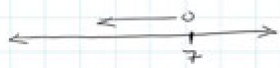
Prove that $\frac{1}{3} < x < 7$.

a) By triangle inequality - sum of 2 sides

Need to check 3 cases:

Case 1 $\quad x+3+3x+6 > 5x+2$

$x < 7$



Case 2 $\quad x+3+5x+2 > 3x+6$

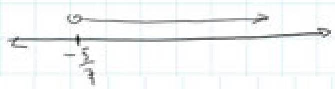
$3x > 1$

$x > 1/3$

Case 3 $\quad 3x+6+5x+2 > x+3$

$7x > -5$

$x > -\frac{5}{7}$



Intersection of all 3

$\frac{1}{3} < x < 7$ as req'd.

2 Provides Correct Solution

1 Correctly uses triangle inequality once to find either $x > 1/3$ or $x < 7$

Marker's comments:
Majority of the students didnt consider all the three cases and lost one mark

(b) Suppose $p \in R$ satisfies $7^p = 2$. Prove that p is irrational. **2**

Assume p is rational.

Since $\quad 7^p = 2$

$\log_7 2 = p$

$= \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$

$7^{\frac{a}{b}} = 2$

$7^a = 2^b$

But since a and b are integers, left hand side is odd whilst right hand side is even, this is a contradiction therefore p is irrational.

2 correct solutions

1 states a, b are positive integers

Marker's comments:
A large number of students didnt recognise that a and b should be positive integers.

(c) Let a_n be the sequence defined recursively by $a_0 = 0$ and $a_n = a_{n-1} + 3n^2$ for all integers $n \geq 1$. **3**

Use mathematical induction to prove that for all integers $n \geq 0$,

$$a_n = \frac{n(n+1)(2n+1)}{2}$$

$a_0 = 0 \quad a_n = a_{n-1} + n^2$

Prove true for $n=0$

ie, $a_0 = \frac{0(0+1)(2\times 0+1)}{2}$

LHS $= a_0$ $= 0$ RHS $= \frac{0\times 1\times 1}{2}$ $= 0$

$\therefore$ LHS $=$ RHS $\therefore$ true for $n=0$ ✓

Assume true for $n=k$, where k is an integer ≥ 0

ie, $a_k = \frac{k(k+1)(2k+1)}{2}$

Prove true for $n=k+1$.

ie, $a_{k+1} = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{2}$

$a_{k+1} = \frac{(k+1)(k+2)(2k+3)}{2}$

LHS $= a_{k+1}$

$= a_k + 3(k+1)^2$ by recursive formula

$= \frac{k(k+1)(2k+1)}{2} + \frac{6(k+1)^2}{2}$ by assumption

$= \frac{(k+1)(k(2k+1)+6(k+1))}{2}$

$= \frac{(k+1)(2k^2+k+6k+6)}{2}$

$= \frac{(k+1)(2k^2+7k+6)}{2}$

$= \frac{(k+1)(k+2)(2k+3)}{2}$

$=$ RHS

$\therefore$ true for $n=k+1$

$\therefore$ Proven true by mathematical induction for $n \geq 0$

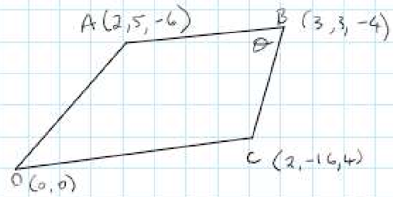
Marker's comments:
Overall , well done

3 correct solution
2 uses recursive formula in inductive step
1 proves true for $n=0$

(d) A quadrilateral is formed in three-dimensional space. **4**

It's vertices are $O(0,0,0)$, $A(2,5,-6)$, $B(3,-3,-4)$ and $C(2,-16,4)$, labelled in the clockwise direction from point O.

Find the size of $\angle ABC$, giving your answer to the nearest degree.



$$\cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$$

$$\overrightarrow{BA} = \begin{bmatrix} 2 \\ 5 \\ -6 \end{bmatrix} - \begin{bmatrix} 3 \\ -3 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -2 \end{bmatrix}$$

$$|\overrightarrow{BA}| = \sqrt{(-1)^2 + 8^2 + (-2)^2} = \sqrt{69}$$

$$\overrightarrow{BC} = \begin{bmatrix} 2 \\ -16 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ -3 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ -13 \\ 8 \end{bmatrix}$$

$$|\overrightarrow{BC}| = \sqrt{(-1)^2 + (-13)^2 + 8^2} = \sqrt{234}$$

$$\cos\theta = \frac{\begin{bmatrix} -1 \\ 8 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -13 \\ 8 \end{bmatrix}}{\sqrt{69} \times \sqrt{234}} = \frac{1 - 104 - 16}{\sqrt{16146}}$$

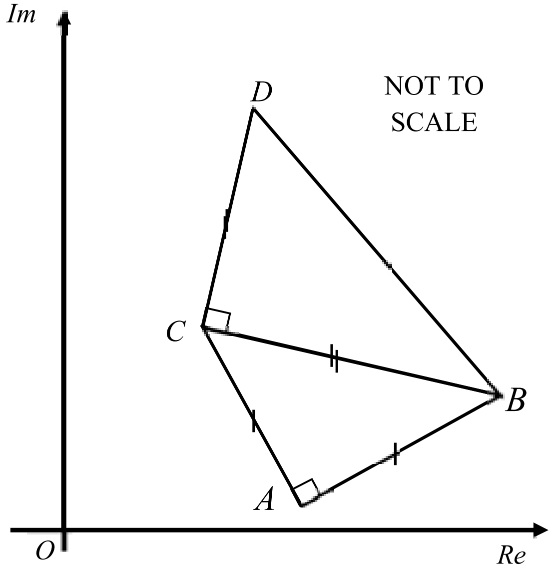
$$\theta = \cos^{-1}\left(\frac{-119}{\sqrt{16146}}\right) = 159.472\ldots$$

$\therefore \angle ABC = 159°$ to nearest degree

4 Correct solution
3 Attempts to calculate $\cos(\angle ABC)$
2 Correct calculation of both $\overrightarrow{BA}$ and $\overrightarrow{BC}$ and $|\overrightarrow{BA}|$ or $|\overrightarrow{BC}|$
1 Finds either $\overrightarrow{BA}$ or $\overrightarrow{BC}$ correctly

Marker's comments:
Well done

(e)



In the diagram the points A, B, C and D represent the complex numbers z_1, z_2, z_3 and z_4, respectively. Both ΔABC and ΔBCD are right angled isosceles triangles as shown

(i) Show that the complex number z_3 can be written as **1**

$$z_3 = (1-i)z_1 + iz_2$$

i) $\vec{AC} = \vec{AB} \times i$

$(z_3 - z_1) = (z_2 - z_1)i$

$z_3 = z_1 + z_2 i - z_1 i$

$\therefore z_3 = (1-i)z_1 + z_2$ as req'd

Marker's comments:
Well done

(ii) Hence, express the complex number z_4 in terms of z_1 and z_2, giving your answer in simplest form. **2**

ii) $\vec{CD} = \vec{CB} \times i$

$z_4 - z_3 = (z_2 - z_3)i$

$z_4 = z_3 - iz_3 + iz_2$

$= (1-i)z_3 + iz_2$

$= (1-i)[(1-i)z_1 + iz_2] + iz_2$

$= (1-i)^2 z_1 + i(1-i)z_2 + iz_2$

$= (1-2i-1)z_1 + z_2 i + z_2 + iz_2$

$= -2iz_1 + 2iz_2 + z_2$

$\therefore z_4 = -2iz_1 + (2i+1)z_2$

Marker's comments:
Some students struggled in showing the understanding of rotation of a vector through ninety degrees anticlockwise

2 Correct solution
1 Makes some progress finding z_4 in terms of z_1 and z_2

Question 14. (14 marks) Use a separate writing booklet.

(a) By considering the series $\sum_{k=1}^{n} k$ and the AM-GM Inequality $\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$, prove that $\left(\frac{n+1}{2}\right)^n \geq n!$ for integers $n \geq 1$. **3**

$$\sum_{k=1}^{n} k = 1+2+3+4 \ldots + n$$

$$\frac{1+2+3+4 \ldots + n}{n} \geq \sqrt[n]{1 \times 2 \times 3 \times \ldots \times n}$$

$1+2+3+\ldots+n$ is an AP

$$S_n = \frac{n}{2}(a+L)$$

$$= \frac{n}{2}(1+n)$$

Hence, $\text{LHS} = \frac{(n+1)\cancel{n}}{2\cancel{n}}$

$$= \frac{n+1}{2}$$

$$1 \times 2 \times 3 \times \ldots n = n!$$

Hence, $\text{RHS} = \sqrt[n]{n!}$

$$\left(\frac{n+1}{2}\right) \geq \sqrt[n]{n!}$$

$$\left[\frac{n+1}{2}\right]^n \geq \left(\sqrt[n]{n!}\right)^n$$

$$\therefore \left(\frac{n+1}{2}\right)^n \geq n! \quad \text{as req'd}$$

Marker's comments:
It is generally a bad idea to use an induction proof unless you have been directed to do so. Any students who used induction could not get any marks as they could not prove the result and did not make any progress towards to correct method.

Responses who received full marks needed to state that $x_1, x_2, \ldots x_n$ was going to the first n positive integers. Many students did not explain how they were using the given AM-GM inequality

3 Correct proof

2 Makes substantial progress by applying the AM-GM inequality and the AP formula or equivalent merit.

1 Makes some progress by applying the AM-GM inequality and the AP formula or equivalent merit

(b) Use integration by parts to evaluate $\displaystyle\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}}\,dx$. **3**

b) Let $I = \displaystyle\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}}\,dx$

$u = \sin^{-1} x \qquad v' = (1+x)^{-\frac{1}{2}}$

$u' = \dfrac{1}{\sqrt{1-x^2}} \qquad v = 2(1+x)^{\frac{1}{2}}$

$$I = 2\sin^{-1}x\sqrt{1+x}\Big]_0^1 - 2\int_0^1 \sqrt{\frac{1+x}{1-x^2}}\,dx$$

$$= 2\left(\tfrac{\pi}{2}\right)\sqrt{2} - 0 - 2\int_0^1 \sqrt{\frac{1+x}{(1-x)(1+x)}}\,dx$$

$$= \pi\sqrt{2} - 2\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$$

$$= \pi\sqrt{2} - 2\left[-2\sqrt{(1-x)}\right]_0^1$$

$$= \pi\sqrt{2} + 4(0-1)$$

$$\therefore I = \pi\sqrt{2} - 4$$

3 Correct solution
2 Makes substantial progress
1 Correctly sets up IBP

Marker's comments:
Relatively well done.

The main error was leaving the negative off when integrating in the third last line.

Lots of very simple algebraic errors and careless **mistakes**

(c) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate **3**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2}{\sin x + 1}\,dx$$

c) $I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2}{\sin x + 1}\,dx$

let $t = \tan\frac{x}{2}$ when $x = \frac{\pi}{3}$ $t = \frac{1}{\sqrt{3}}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ $x = \frac{\pi}{2}$ $t = 1$

$= \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)$ $\sin x = \frac{2t}{1+t^2}$

$= \frac{1+t^2}{2}$

$dx = \frac{2}{1+t^2}\,dt$

$$I = 2\int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{\frac{2t}{1+t^2} + 1} \frac{dt}{1+t^2}$$

$$= 4\int_{\frac{1}{\sqrt{3}}}^{1} \frac{1+t^2}{t^2+2t+1} \times \frac{dt}{1+t^2}$$

$$= 4\int_{\frac{1}{\sqrt{3}}}^{1} \frac{dt}{(t+1)^2}$$

$$= -4\left[\frac{1}{t+1}\right]_{1/\sqrt{3}}^{1}$$

$$= -4\left[\frac{1}{2} - \frac{1}{\frac{1}{\sqrt{3}}+1}\right]$$

$$= -4\left[\frac{1}{2} - \frac{\sqrt{3}}{1+\sqrt{3}}\right]$$

$$= -4\left[\frac{1+\sqrt{3}-2\sqrt{3}}{2(1+\sqrt{3})}\right]$$

$$= -2\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)$$

$$\therefore I = 4 - 2\sqrt{3}$$

Marker's comments:
Relatively well done

Some very elaborate ways of calculating ' $\frac{dx}{dt}$ in terms of t. The easiest is to make x the subject and $x = 2\tan^{-1} t$ differentiate. Lots of wasted time on this section of the question.

There were many different forms of the solution, depending on whether the denominator was rationalised.

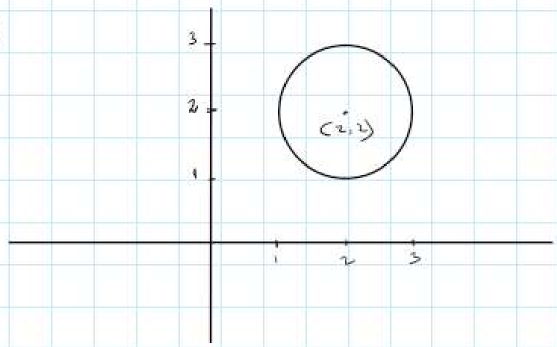
3 Correct solution
2 Correctly applies t substitution and the limits of integration
1 Attempts to use t substitution

(d) Given $|z-2-2i|=1$.

(i) On an argand diagram, sketch the graph of the set of points represented by z. **1**

$|z-2-2i| = 1$

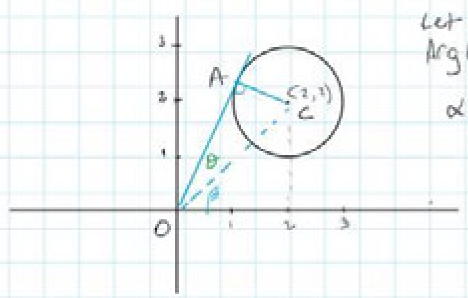
i)



Marker's comments:
Very well done

(ii) Find the maximum value of Arg z. **2**

Maximum argument occurs when the tangent is on left side of circle



Let
$\text{Arg}(z) = \alpha$

$\alpha = \theta + \beta$

$\beta = \tan^{-1}(1)$
$= \frac{\pi}{4}$

$|OC| = 2\sqrt{2}$

$\sin\theta = \frac{1}{2\sqrt{2}}$ $\therefore \theta = \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

$\alpha = \frac{\pi}{4} + \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

$\therefore \text{Arg}(z) = 65° 42'$

Marker's comments:
Quite poorly done.

Many students did not know how to do this.

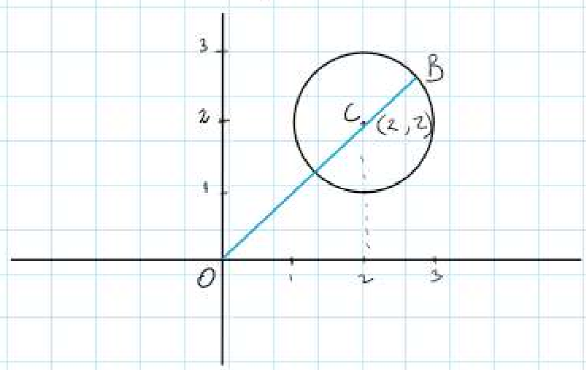
Some students used the correct method but put the right angle at the centre of the circle instead of between the tangent and radius.

Some students assumed the the max arg was at the point (1,2). This received no marks

2 Correct solution

1 Recognises tangent to left of circle provides maximum argument or equivalent merit

(iii) Find the maximum value of $|z|$. **2**



Maximum value of $|z|$ is at B

$\max |z| = |\overrightarrow{OC}| + 1$

$\therefore \max |z| = 2\sqrt{2} + 1$

Marker's comments:
Much better

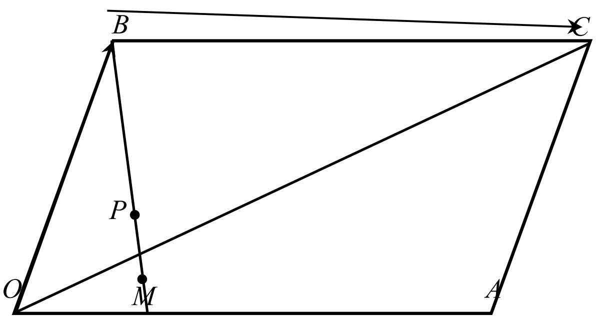
2 Correct solution

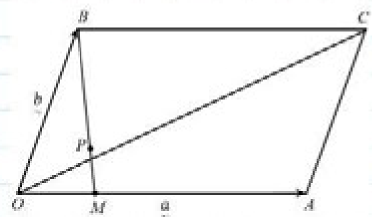
1 Recognises max $|z|$ goes through the centre to the other side of circle

Question 15. (16 marks) Use a separate writing booklet.

(a) $OACB$ is a parallelogram with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. M is a point on OA such that $\left|\overrightarrow{OM}\right| = \frac{1}{5}\left|\overrightarrow{OA}\right|$. P is a point on MB such that $\left|\overrightarrow{MP}\right| = \frac{1}{6}\left|\overrightarrow{MB}\right|$, as shown in the diagram. **3**

Show that P lies on OC.





$\left|\overrightarrow{OM}\right| = \frac{1}{5}\left|\overrightarrow{OA}\right|$

$\left|\overrightarrow{MP}\right| = \frac{1}{6}\left|\overrightarrow{MB}\right|$

$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$

$= \frac{1}{5}\underset{\sim}{a} + \frac{1}{6}\underset{\sim}{b} - \frac{1}{30}\underset{\sim}{a}$

$= \frac{5}{30}\underset{\sim}{a} + \frac{1}{6}\underset{\sim}{b}$

$\therefore \overrightarrow{OP} = \frac{1}{6}(\underset{\sim}{a} + \underset{\sim}{b})$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= \underset{\sim}{a} + \underset{\sim}{b}$

$\therefore$ Since $\overrightarrow{OP}$ is a scalar multiple of $\overrightarrow{OC}$, then it must lie on $\overrightarrow{OC}$.

$\overrightarrow{MP} = \frac{1}{6}\overrightarrow{MB}$

$\overrightarrow{MB} = \overrightarrow{OB} - \overrightarrow{OM}$

$= \overrightarrow{OB} - \frac{1}{5}\overrightarrow{OA}$

$= \underset{\sim}{b} - \frac{1}{5}\underset{\sim}{a}$

$\overrightarrow{MP} = \frac{1}{6}\left(\underset{\sim}{b} - \frac{1}{5}\underset{\sim}{a}\right)$

$= \frac{1}{6}\underset{\sim}{b} - \frac{1}{30}\underset{\sim}{a}$

Marker's comments:
Very well done

3 Correct solution
2 Attempts to write $\overrightarrow{OP}$ as a multiple of $\overrightarrow{OC}$ or equivalent merit.
1 Writes $\overrightarrow{OP}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{MB}$ or equivalent merit.

(b) (i) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **1**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

let $u = a - x$ when $x=0$ $u=a$

$x = a - u$ $x=a$ $u=0$

$dx = -du$

$$\int_0^a f(x)dx = \int_a^0 f(a-u) - du$$

$$= -\int_a^0 f(a-u)\,du$$

$$= \int_0^a f(a-u)\,du$$ replacing u with x since they're dummy variables

$$= \int_0^a f(a-x)dx$$ as req'd

Marker's comments:
Generally well done.

Make sure you don't have two variables in the definite integral

(ii) Show that: **2**

$$\int_0^1 \frac{dx}{x+\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}d\theta$$

$$\int_0^1 \frac{dx}{x+\sqrt{1-x^2}}$$

let $x = \sin\theta$

$dx = \cos\theta \; d\theta$

$x=0$ $\theta=0$

$x=1$ $\theta = \pi/2$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \sqrt{1-\sin^2\theta}}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta$$ as req'd

2 Correct solution

1 Applies suitable substitution and correct limits

Marker's comments:
Very well done

This is a show question and so you must show the substitution and then simplify.

(iii) Hence, determine the value of: **2**

$$I = \int_0^1 \frac{dx}{x+\sqrt{1-x^2}}$$

$$\int_0^1 \frac{dx}{x+\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta$$

let $I_1 = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\left(\frac{\pi}{2}-\theta\right)}\,d\theta \quad \{\text{from part(i)}\}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta+\sin\theta}\,d\theta$$

$$I_1 + I_2 = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta} + \frac{\sin\theta}{\cos\theta+\sin\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta+\sin\theta}{\sin\theta+\cos\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 1\,d\theta$$

$$= \left[\theta\right]_0^{\frac{\pi}{2}}$$

$I_1 + I_2 = \frac{\pi}{2}$ But $I_1 = I_2$, hence

$$2I = \frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^1 \frac{dx}{x+\sqrt{1-x^2}} = \frac{\pi}{4}$$

2 Correct solution

1 Attempts to use the symmetry from part (i)

Marker's comments:
Very well done
Some students did not make the connection with part i) and did not know how to start.

(c) The line $\underset{\sim}{r}_1$ has equation:

$$\underset{\sim}{r}_1 = \underset{\sim}{i} + 2\underset{\sim}{k} + \lambda\left(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}\right) \quad \text{where } \lambda \in \mathbb{R}.$$

The line $\underset{\sim}{r}_2$ has equation:

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-1}{3}$$

(i) Show that $\underset{\sim}{r}_2 = -\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k} + \mu\left(\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$ where $\mu \in \mathbb{R}$. **1**

$\mu = \dfrac{x+1}{1} = \dfrac{y-4}{1} = \dfrac{z-1}{3}$

$x = -1 + \mu \quad y = 4 + \mu \quad z = 1 + 3\mu$

$\therefore \underset{\sim}{r}_2 = -\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k} + \mu\left(\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$ as req'd

Marker's comments:
Lots of students did not know that the equation of a line in the form $\dfrac{x+1}{1} = \dfrac{y-4}{1} = \dfrac{z-1}{3}$.

Many just stated the direction vector without showing it.

A lot of unnecessary long working with this question.

(ii) Show that lines $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ do not intersect. **3**

From $\underset{\sim}{r}_1$:

$x = 1 + 2\lambda \quad y = 3\lambda \quad z = 2 - \lambda$

$-1 + \mu = 1 + 2\lambda$ — (1)

$4 + \mu = 3\lambda$ — (2)

$1 + 3\mu = 2 - \lambda$ — (3)

from (1) $\mu = 2 + 2\lambda$ sub into (2)

$4 + 2 + 2\lambda = 3\lambda \Rightarrow \lambda = 6$

$\mu = 14$

sub $\lambda = 6$ and $\mu = 14$ into (3)

LHS $= 1 + 3 \times 14$ $= 43$ RHS $= 2 - 6$ $= -4$

LHS $\neq$ RHS

$\therefore \underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ do not intersect

Marker's comments:
Generally very well done

3 Correct solution

2 Attempts to show contradiction

1 Attempts to solve a pair of equations to find at least one of either λ or μ

The point A lies on $\underset{\sim}{r}_1$ with parameter $\lambda = p$, and the point B lies on $\underset{\sim}{r}_2$ with parameter $\mu = q$.

(iii) Write $\overrightarrow{AB}$ as a column vector. **1**

Point A : $\begin{bmatrix} 1+2p \\ 3p \\ 2-p \end{bmatrix}$ B : $\begin{bmatrix} -1+q \\ 4+q \\ 1+3q \end{bmatrix}$

$$\overrightarrow{AB} = \begin{bmatrix} -1+q-1-2p \\ 4+q-3p \\ 1+3q-2+p \end{bmatrix}$$

$$= \begin{bmatrix} -2+q-2p \\ 4+q-3p \\ -1+3q+p \end{bmatrix}$$

Marker's comments:
Generally very well done

(iv) Calculate the value of $\left|\overrightarrow{AB}\right|$ when $\overrightarrow{AB}$ is perpendicular to both $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$. **3**

Since $\overrightarrow{AB}$ is perpendicular to both $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ then

$$\begin{bmatrix} -2+q-2p \\ 4+q-3p \\ -1+3q+p \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = 0$$

$$-4+2q-4p+12+3q-9p+1-3q-p=0$$

$$2q-14p=-9 \quad \text{—— (1)}$$

$$\begin{bmatrix} -2+q-2p \\ 4+q-3p \\ -1+3q+p \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 0$$

$$-2+q-2p+4+q-3p-3+9q+3p=0$$

$$11q-2p=1 \quad \text{—— (2)}$$

Marker's comments:
Generally well done

3 Correct solution
2 Correct calculation of p and q
1 Attempts to calculate the scalar product between $\overrightarrow{AB}$ and either $\underset{\sim}{r}_1$ or $\underset{\sim}{r}_2$

(2) $\times 7$

$77q - 14p = 7$ — (3)

$2q - 14p = -9$ — (1)

$75q = 16$

$\therefore q = \frac{16}{75} \quad p = \frac{101}{150}$

$\therefore \overrightarrow{AB} = \begin{bmatrix} -\frac{47}{15} \\ \frac{329}{150} \\ \frac{47}{150} \end{bmatrix}$

$|\overrightarrow{AB}| = \sqrt{\left(-\frac{47}{15}\right)^2 + \left(\frac{329}{150}\right)^2 + \left(\frac{47}{150}\right)^2}$

$\therefore |\overrightarrow{AB}| = \frac{47\sqrt{6}}{30}$

Question 16. (16 marks) Use a separate writing booklet.

(a) Consider the proposition:

If the remainder is 2 or 3 when an integer n *is divided by 4, then* $n \neq k^2$*, where* $k \in \mathbb{Z}$.

(i) State the contrapositive to the proposition. **1**

P : remainder is 2 or 3 when an integer, n, is divided by 4

Q : $n \neq k^2$

$P \Rightarrow Q$

Contrapositive is $\sim Q \Rightarrow \sim P$

If $n = k^2$, when $k \in \mathbb{Z}$, then the remainder is 0 or 1 when an integer, n, is divided by 4.

{ Can also be written as if $n = k^2$, where $k \in \mathbb{Z}$, then the remainder is not 2 or 3 when an integer, n, is divided by 4 }

$n = k^2$

Marker's comments:
Well done

(ii) Hence, prove the proposition by proving the contrapositive. **3**

$n = k^2$

Case 1 $\quad k = 4q$ when $q \in \mathbb{Z}$

$k^2 = 16q^2$

$n = 16q^2$ remainder is 0 when divided by 4

Case 2 $\quad k = 4q + 1$

$k^2 = (4q+1)^2$

$= 16q^2 + 8q + 1$

$n = 4(4q^2 + 2q) + 1$ remainder is 1 when divided by

Case 3 $\quad k = 4q + 2$

$k^2 = (4q+2)^2$

$= 16q^2 + 16q + 4$

$n = 4(4q^2 + 4q + 1)$ remainder is 0 when divided by 4

Case 4 $\quad k = 4q + 3$

$k^2 = (4q+3)^2$

$= 16q^2 + 24q + 9$

$n = 4(4q^2 + 6q + 2) + 1$ remainder is 1 when divided by 4

$\therefore$ Since the contrapositive statement is true, then the original statement is true

3 Correct proof
2 Sets up cases and makes some progress or equivalent ment
1 Sets up k, including definition of q or equivalent ment.

Marker's comments:
The students are reminded to read the question carefully especially in these questions when k was any integer .

(b) Let

$$z_n = \frac{1}{(1+i)^0} + \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^n}$$

(i) Express $\frac{1}{1+i}$ in the form $x+iy$. **1**

$$\frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{2}$$

$$\therefore \frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i$$

Marker's comments:
Well done

(ii) Prove that $z_n = 1 - i + \dfrac{\sin\frac{\pi n}{4} + i\cos\frac{\pi n}{4}}{2^{\frac{n}{2}}}$. **4**

Given $z_n = \frac{1}{(1+i)^0} + \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^n}$

z_n is a G.P. with

$a = 1 \quad r = \frac{1}{1+i} = \frac{1}{2}(1-i)$ from part (i)

number of terms $= n+1$

$$S_n = \frac{a(1-r^{n+1})}{1-r}$$

$$= \frac{1\left(1-\left(\frac{1-i}{2}\right)^{n+1}\right)}{1-\left(\frac{1-i}{2}\right)}$$

$$= \frac{2^{n+1}-(1-i)^{n+1}}{2^{n+1}} \times \frac{2}{1+i}$$

$$= \frac{2^{n+1}-(1-i)^{n+1}}{2^n} \times \frac{1}{1+i}$$

Marker's comments:
Poorly done . Some students didnt take the correct number of terms and others struggled in the last steps of the solution .

4 Correct solution
3 Makes significant progress tow's solution
2 Use De Moivre's theore and has some progress
1 Use geometric sum formula correctly, or equivalent merit

$$= \left[\frac{2^{n+1} - (1-i)^{n+1}}{2^n}\right] \times \frac{(1-i)}{2}$$

$$= \frac{(2^{n+1} - (1-i)^{n+1})(1-i)}{2^{n+1}}$$

$$= \frac{2^{n+1}(1-i)}{2^{n+1}} - \frac{(1-i)^{n+2}}{2^{n+1}}$$

$$= 1 - i - \frac{1}{2^{n+1}}(1-i)^{n+2}$$

$$= 1-i - \frac{1}{2^{n+1}}\left[\sqrt{2}\,\mathrm{cis}\left(\frac{-\pi}{4}\right)\right]^{n+2}$$

$$= 1-i - \frac{1}{2^{n+1}}\left[(2^{\frac{1}{2}})^{n+1}\,\mathrm{cis}\left(\frac{-(n+2)\pi}{4}\right)\right] \text{ by D.M.T}$$

$$= 1-i - \frac{1}{2^{\frac{n}{2}}}\,\mathrm{cis}\left(-\frac{n\pi}{4} - \frac{\pi}{2}\right)$$

$$= 1-i - \frac{1}{2^{n/2}}\left[\cos\left(\frac{-n\pi}{4} - \frac{\pi}{2}\right) + i\sin\left(\frac{-n\pi}{4} - \frac{\pi}{2}\right)\right]$$

$$= 1-i - \frac{1}{2^{\frac{n}{2}}}\left[\cos\left(\frac{-n\pi}{4}\right)\cos\frac{\pi}{2} + \sin\left(\frac{-n\pi}{4}\right)\sin\frac{\pi}{2} + i\sin\left(\frac{-n\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - i\sin\frac{\pi}{2}\cos\left(\frac{-n\pi}{4}\right)\right]$$

$$= 1-i - \frac{1}{2^{\frac{n}{2}}}\left[0 - \sin\frac{n\pi}{4} + 0 - i\cos\left(\frac{-n\pi}{4}\right)\right]$$

$$= 1-i - \frac{1}{2^{n/2}}\left(-\sin\left(\frac{n\pi}{4}\right) - i\cos\left(\frac{n\pi}{4}\right)\right]$$

$$= 1-i + \frac{1}{2^{\frac{n}{2}}}\left(\sin\left(\frac{n\pi}{4}\right) + i\cos\left(\frac{n\pi}{4}\right)\right] \text{ as req'd.}$$

(c) Let $I_n = \int_0^a x^n \sqrt{a^2 - x^2}\, dx$, $a \in R^+$ and $n = 0, 1, \ldots$

(i) Prove that $I_n = a^2 \dfrac{n-1}{n+2} I_{n-2}$ for $n = 2, 3, \ldots$ **3**

$$I_n = \int_0^a x^{n-1} \times x \sqrt{a^2 - x^2}\, dx$$

let $u = x^{n-1}$ $\quad v' = x\sqrt{a^2 - x^2}$

$u' = (n-1)x^{n-2}$ $\quad v = \frac{-2}{6}(a^2 - x^2)^{3/2}$

$$I_n = \left. \frac{-2}{6} x^{n-1}(a^2 - x^2)^{\frac{3}{2}} \right]_0^a + \frac{2(n-1)}{6} \int_0^a x^{n-2}(a^2 - x^2)^{\frac{3}{2}}\, dx$$

$$= 0 + 0 + \frac{2(n-1)}{6} \int_0^a x^{n-2}(a^2 - x^2)^{\frac{3}{2}}\, dx$$

$$= \frac{2(n-1)}{6} \int_0^a x^{n-2}(a^2 - x^2)^{\frac{1}{2}} \times (a^2 - x^2)\, dx$$

$$= \frac{2(n-1)}{6} \int_0^a a^2 x^{n-2}\sqrt{(a^2 - x^2)} - x^n \sqrt{a^2 - x^2}\, dx$$

$$= \frac{2(n-1)}{6} \int_0^a a^2 x^{n-2} \sqrt{a^2 - x^2}\, dx - \frac{2(n-1)}{6} \int_0^a x^n \sqrt{a^2 - x^2}\, dx$$

$$= \frac{2(n-1)}{6} a^2 I_{n-2} - \frac{2(n-1)}{6} I_n$$

$$I_n + \frac{2}{6}(n-1) I_n = a^2 \frac{2(n-1)}{6} I_{n-2}$$

$$I_n \left(1 + \frac{n}{3} - \frac{1}{3}\right) = a^2 \frac{(n-1)}{3} I_{n-2}$$

$$I_n \left(\frac{2+n}{3}\right) = \frac{a^2(n-1)}{3} I_{n-2}$$

$$I_n = a^2 \frac{(n-1)}{n+2} I_{n-2} \quad \text{as req'd}$$

Marker's comments:
Majority of the students got u and v correct but lost marks due to silly mistakes .

3 Correct proof

2 Correctly applies integration by parts and attempts to reduce to reduce to integral, or equivalent merit.

1 Attempts to use integration by parts or equivalent merit.

(ii) Prove that $I_{2n} = \pi\left(\frac{a}{2}\right)^{2n+2} \frac{(2n)!}{n!(n+1)!}$ **4**

$I_{2n} = \frac{a^2(2n-1)}{2n+2} I_{2n-2}$

$I_{2n-2} = \frac{a^2(2n-3)}{2n} I_{2n-4}$

$I_{2n-4} = \frac{a^2(2n-5)}{2n-2} I_{2n-6}$

$= \frac{a^2(2n-1)}{2n+2}\left(\frac{a^2(2n-3)}{2n} I_{2n-4}\right)$

$= \frac{a^2(2n-1)}{2n+2}\left(\frac{a^2(2n-3)}{2n}\right)\left(\frac{a^2(2n-5)}{2n-2} I_{2n-6}\right)$

$= (a^2)^3 \frac{(2n-1)(2n-3)(2n-5)}{(2n+2)\,2n\,(2n-2)} I_{2n-6}$

$\therefore I_{2n} = (a^2)^n \frac{(2n-1)(2n-3)(2n-5)\cdots 1}{(2n+2)\,2n\,(2n-2)\cdots 4} I_0$

$= (a^2)^n \frac{(2n-1)(2n-3)(2n-5)\cdots 1}{(2n+2)\,2n\,(2n-2)\cdots 4} I_0 \times \frac{2n\,(2n-2)(2n-4)\cdots(4)(3)(2)(1)}{2n\,(2n-2)(2n-4)\cdots(4)(4)(2)}$

$= (a^2)^n \frac{2n\,(2n-1)(2n-2)(2n-3)\cdots 1}{(2n+2)(2n)(2n)(2n-2)(2n-2)\cdots(2)(2)(1)} I_0$

$= (a^2)^n \frac{(2n)!}{2(n+1)\,2(n)\,2(n)\,2(n-1)\,2(n-1)\cdots(2)(2)(1)} I_0$

$= (a^2)^n \frac{(2n)!}{2^{2n}\,n!\,(n+1)!} I_0$

$\therefore I_{2n} = \left(\frac{a}{2}\right)^{2n} \frac{(2n)!}{n!\,(n+1)!} I_0$

$I_0 = \int_0^a x^0 \sqrt{a^2-x^2}\,dx$

$= \int_0^a \sqrt{a^2-x^2}\,dx$ is area of quarter of a circle of radius a.

Hence, $I_0 = \frac{\pi a^2}{4}$

Thus, $I_{2n} = \left(\frac{a}{2}\right)^{2n} \frac{(2n)!}{n!\,(n+1)!} \times \frac{\pi a^2}{4}$

$= \pi\left(\frac{a}{2}\right)^{2n} \times \frac{a^2}{2^2} \times \frac{(2n)!}{n!\,(2n+1)!}$

$\therefore I_{2n} = \pi\left(\frac{a}{2}\right)^{2n+2} \times \frac{(2n)!}{n!\,(2n+1)!}$ as req'd

Marker's comments:
Not many students received full marks as they either struggled with the time management and couldnt complete the question or couldnt get to the final part of the solution due to lack of understanding in factorials

4 Correct proof
3 Identifies I_0 as area of quarter circle or equivalent merit
2 Makes substantial progress and correctly derives $\left(\frac{a}{2}\right)^{2n} \frac{(2n)!}{n!\,(n+1)!} I_0$
1 Attempts to setup pattern or equivalent merit